WINGS IN HYPERSONIC FLOW

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In the solution of many applied problems of the mechanics of a continuous medium, reducing to systems of partial differential equations, the methods of integral relations have found a wide application. These methods make it possible, in an approximate solution of the problems, to decrease the number of independent variables in the differential equations and even to reduce these equations to algebraic ones.

Great popularity has been achieved in the course of fifty years by the method of B.G.Galerkin. As is well known, in Galerkin's method the form of the solution is chosen a priori, whilst the integral relations, turning into algebraic equations, serve to determine the constants appearing in the solution. Kantorovich [1] proposed in problems with two variables to seek a solution in a form containing undetermined functions of one variable, and to determine these functions from the ordinary differential equations obtained from the integral relations. In an important particular problem of fluid mechanics — the theory of the boundary layer — such an approach had been employed earlier in the method of integral relations by von Kármán [2].

In a number of problems the method of integral relations enables one to obtain good results with a very small number of approximations and even in the first approximation. For this a considerable importance attaches to the a priori choice of the particular stipulated solution, based on the use of supplementary information on the form of the required solution (as examples we may cite the Kochin-Loitsianskii method in boundary layer theory [2] or the method used by the author in the calculation of one-dimensional unsteady gas flows with strong shock waves [3]). The application of high speed computers makes it possible to effectively find sufficiently high approximations in the method of integral relations and at the same time makes it possible to relax the requirements in the a priori choice of the specified part of the solution and the form of the original equations. However, the use of high approximations complicates the qualitative analysis of the solution of the approximating system of equations and the interpretation of the results obtained. In the present paper the method of integral relations is applied to three-dimensional gas flows with shock waves. We make a qualitative analysis of the system of equations of the approximation of zero order, and these equations are interpreted as the equations of two-dimensional motion of gas on a streamline surface.

Let us turn first to the basic idea of the method of integral relations. Let us consider a system of n first order partial differential equations relating to the functions $u_1, \ldots u_n$ of the three independent variables x_1, x_2, z

$$L_i(u) = 0 \qquad (i = 1, 2, \ldots, n)$$

Suppose that it is required to find the solution of this system in the region D, and we shall assume for definiteness that the region D is bounded by the surface $\chi(x_1, x_2) = 0$ and the surfaces $z = z_w(x_1, x_2)$, $z = z_s(x_1, x_2)$.

We shall assume that the boundary conditions have the form $\psi_k(u, x_1, x_2) = 0$ on the surfaces $z = z_w$ and $z = z_s$, and $\psi_l^*(u, x_1, x_2, z) = 0$ on the surface $\chi = 0$ (depending on the nature of the problem the latter conditions can also be different).

Suppose that we have succeeded in finding the system of functions $\varphi_n(x_1, x_2, z)$ possessing the property that any function $u(x_1, x_2, z)$, continuous in the region D, can be approximated by a certain linear combination of the functions of this system.

We shall expand the approximate solution of the problem in the form

$$u_k^{(N)} = \sum_{m=0}^{N} u_{km}^{(N)}(x_1, x_2)\,\varphi_m(x_1, x_2, z)$$

To determine the coefficients $u_{km}^{(N)}$ involves taking the required number of integral relations (the conditions of orthogonality of the expression $L_i(u^{(N)})$ with the functions ψ_m)

$$\int_{z_w}^{z_s} L_i(u^{(N)})\,\psi_m(x_1, x_2, z)\,dz = 0 \qquad (m = 0, 1, \ldots)$$

where $\psi_n(x_1, x_2, z)$ is a system of functions, complete in the region D (in particular, the systems of functions φ_n and ψ_n can be coincident), and also the boundary conditions $\psi_k(u^{(N)}, x_1, x_2) = 0$, giving closed relations between the functions $u_{km}^{(N)}$. The integral relations turn into first order partial differential equations for the functions $u_{km}^{(N)}$ in two independent variables. The solution of these equations is expanded on the region of the (x_1, x_2) plane bounded by the curve $\chi(x_1, x_2) = 0$. If the boundary conditions on the boundary of the region had the form $\psi_l^*(u, x_1, x_2, z) = 0$, then we should find the required conditions for the two-dimensional problem from the relations

$$\int_{z_w}^{z_s} \psi_l^*(u^{(N)}, x_1, x_2, z)\,\psi_m(x_1, x_2, z)\,dz = 0$$

The formulation of the problem is generalized without difficulty to the case when, for example, the surface $z = z_s(x_1, x_2)$ is not given but determines itself. Then the conditions on the surface $z = z_s$ take the form

$$\psi_k(u, x_1, x_2, z, \partial z_s/\partial x_1, \partial z_s/\partial x_2) = 0$$

and the number of them increases by one.

For the choice of the functions φ_n and ψ_n the following general method can be recommended. Let $\eta_n(z)$ be a system of linearly independent functions, complete on the segment $[a, b]$.

Then the system of functions $\eta_n(\zeta)$, where

$$\zeta = \frac{(b-a)z + az_s - bz_w}{z_s - z_w}$$

is complete on the segment $[z_w, z_s]$. Therefore in the region D we can use the system of functions $\eta_n(\zeta)$ for φ_n and ψ_n.

A distinctive choice of the orthogonal functions ψ_n was proposed by Dorodnitsyn [4]. The functions ψ_n start off being dependent on the chosen approximation and are determined from Formulas

$$\psi_m^{(N)} = \begin{cases} 1 & \text{for } 0 < \zeta < (m+1)/N \\ 0 & \text{for } (m+1)/N < \zeta < 1 \end{cases} \qquad (m = 0, 1, \ldots, N-1)$$

For the functions φ_n used for the approximation to the solution, we take the power functions ζ^n. Using the choice of the functions φ_n and ψ_n, Belotserkovskii gave an effective numerical solution of a number of problems of two-dimensional flow past bodies with shock waves present [5].

1. The general equations. Let us apply the above general considerations to the problem of supersonic streamline flow of an ideal gas past a body. For simplicity we shall assume that the portion of the surface of the body under consideration is plane (in particular, it can be assumed that the case in question concerns the flow past a plane wing at an angle of attack). To describe the motion of the gas let us introduce Cartesian coordinates, choosing the axes of x and y to lie in the plane of the body surface, whilst the z-axis is directed along the normal to it.

The equations of motion of the gas in the layer between the surface of the wing and the shock wave will be taken in the form

$$\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0, \qquad \frac{\partial}{\partial x}(\rho u^2+p)+\frac{\partial \rho uv}{\partial y}+\frac{\partial \rho uw}{\partial z}=0$$

$$\frac{\partial \rho vu}{\partial x}+\frac{\partial}{\partial y}(\rho v^2+p)+\frac{\partial \rho vw}{\partial z}=0, \qquad \frac{\partial \rho wu}{\partial x}+\frac{\partial \rho wv}{\partial y}+\frac{\partial}{\partial z}(\rho w^2+p)=0 \quad (1.1)$$

$$\frac{\partial \rho uS}{\partial x}+\frac{\partial \rho vS}{\partial y}+\frac{\partial \rho wS}{\partial z}=0, \qquad \frac{\partial \rho ui^*}{\partial x}+\frac{\partial \rho vi^*}{\partial y}+\frac{\partial \rho wi^*}{\partial z}=0$$

Here u, v, w are the velocity components along the axes; ρ, p, S, i^* are the density, pressure, entropy and total enthalpy of unit mass of the gas, respectively. For a perfect gas with constant specific heats

$$S=\frac{p^{1/\gamma}}{\rho}, \qquad i^*=\frac{u^2+v^2+w^2}{2}+\frac{\gamma}{\gamma-1}\frac{p}{\rho}$$

The last equation of the system (1.1), expressing the conservation of total enthalpy in a particle of the gas, is not independent – it is obtained as a result of the remaining equations of the system.

The system of equations (1.1) can be rewritten in the general form

$$\frac{\partial A_{ij}}{\partial x_j}+\frac{\partial B_i}{\partial z}=0 \qquad (1.2)$$

Here

$$x_1=x, \qquad x_2=y$$

$$A_{11}=\rho u, \quad A_{12}=\rho v, \quad B_1=\rho w, \quad A_{21}=\rho u^2+p, \quad A_{22}=\rho uv, \quad B_2=\rho uw,$$

$$A_{31}=\rho vu, \quad A_{32}=\rho v^2+p, \quad B_3=\rho vw, \quad A_{41}=\rho wu, \quad A_{42}=\rho wv, \quad B_4=\rho w^2+p$$

$$A_{51}=\rho uS, \quad A_{52}=\rho vS, \quad B_5=\rho wS, \quad A_{61}=\rho ui^*, \quad A_{62}=\rho vi^*, \quad B_6=\rho wi^*$$

Let $h=h(x,y)$ be the thickness of the layer of gas between the surface of the wing and the shock wave . On the shock wave, i.e. when $z=h(x,y)$, there must be fulfilment of the conditions of conservation of mass, momentum (in projection on the three axes) and of total enthalpy. These conditions in the notation introduced above can be written down in the following form:

$$\left(B_i-A_{ij}\frac{\partial h}{\partial x_j}\right)_{z=h}=B_i^{\infty}-A_{ij}^{\infty}\frac{\partial h}{\partial x_j} \qquad (i=1,2,3,4,6) \qquad (1.3)$$

The superscript ∞ here denotes values in the free stream.

When $i=5$, i.e. for entropy, the conservation law does not hold at the shock wave, as is well known. The value of the entropy behind the shock wave S_h is expressed with the help of the equations of conservation (1.3) in terms of the parameters of the free stream and $\partial h/\partial x_j$. For a perfect gas with constant specific heats

$$S_h = \frac{1}{\rho^\infty}\left[\frac{2}{(\gamma+1)\rho^\infty}\,\frac{(B_1 - A_{1j}\,\partial h/\partial x_j)^2}{1+(\partial h/\partial x_j)(\partial h/\partial x_j)} - \frac{\gamma-1}{\gamma+1}p^\infty\right]^{1/\gamma}\times$$
$$\times\left[\frac{\gamma-1}{\gamma+1} + \frac{2\gamma}{\gamma+1}p^\infty\rho^\infty\,\frac{1+(\partial h/\partial x_j)(\partial h/\partial x_j)}{(B_1 - A_{1j}\,\partial h/\partial x_j)^2}\right] \tag{1.4}$$

Making use of this expression, we can write for $i = 5$ also

$$\left(B_5 - A_{5j}\frac{\partial h}{\partial x_j}\right)_{z=h} = B_5^{\infty} - A_{5j}^{\infty}\frac{\partial h}{\partial x_j} \tag{1.5}$$

Here we have introduced the conventional notation $B_5^\infty = B_1^\infty S_h$, $A_{5j}^\infty = A_{1j}^\infty S_h$.

We shall introduce, in accordance with the usual theory, instead of the coordinate z the variable ζ according to Formula

$$\zeta = 2z/h - 1$$

The variable ζ ranges from -1 to $+1$, and $\zeta = -1$ corresponds to the plane of the wing, whilst $\zeta = +1$ corresponds to the shock wave surface. For the functions φ_n we take the Legendre polynomials $P_n(\zeta)$, forming a complete system of orthogonal functions in the interval $[-1, +1]$, and we shall approximate the required functions, for example the function u, by expressions of the form

$$u^{(N)} = \sum_{m=0}^{N} P_m(\zeta)\,u_m^{(N)}$$

which we shall call the Nth approximation for these functions.

For the orthogonalizing functions ψ_n we shall likewise take the Legendre polynomials $P_n(\zeta)$. Let us multiply term by term the equations of system (1.2) by the Legendre polynomial of the mth order $P_m(\zeta)$ and integrate them with respect to z from 0 to h

$$\int_0^h P_m(\zeta)\frac{\partial A_{ij}}{\partial x_j}dz + \int_0^h P_m(\zeta)\frac{\partial B_i}{\partial z}dz = 0$$

Carrying out a simple transformation and making use of the properties of Legendre polynomials, we obtain

for $m = 0$

$$\frac{\partial}{\partial x_j}\frac{h}{2}\int_{-1}^{1} A_{ij}\,d\zeta + \left(B_i - A_{ij}\frac{\partial h}{\partial x_j}\right)_{z=h} - (B_i)_{z=0} = 0$$

for $m = 1, 2, 3, \ldots$

$$\frac{\partial}{\partial x_j}\frac{h}{2}\int_{-1}^{1} P_m A_{ij}\,d\zeta + \frac{1}{2}\frac{\partial h}{\partial x_j}\int_{-1}^{1}\left[mP_m + \sum_{k=1}^{m}(2m-2k+1)P_{m-k}\right]A_{ij}d\zeta -$$

(to be continued)

$$-\int_{-1}^{1} B_i \sum_{k=1}^{n} (2m-4k+3)\, P_{m-2k+1}\, d\zeta + \left(B_i - A_{ij}\frac{\partial h}{\partial x_j}\right)_{z=h} - (B_i)_{z=0} = 0$$

$$\left(n = \frac{m}{2},\ \frac{m+1}{2} \text{ respectively for } m \text{ even or odd}\right)$$

Using the conditions (1.3) and (1.5) on the shock wave and the condition $w(x_1, x_2, 0) = 0$ on the streamlined surface, we can rewrite this system of integral relations in the following form $(i = 1, \ldots, 6)$:

$$\frac{\partial}{\partial x_j}\frac{h}{2}\int_{-1}^{1} A_{ij}\, d\zeta + B_i^{\infty} - A_{ij}^{\infty}\frac{\partial h}{\partial x_j} - (B_i)_{z=0} = 0 \qquad (m=0)$$

$$\frac{\partial}{\partial x_j}\frac{h}{2}\int_{-1}^{1}\left(P_m - P_{m-1}\right) A_{ij}\, d\zeta + \frac{1}{2}\frac{\partial h}{\partial x_j}\int_{-1}^{1} m\left(P_m + P_{m-1}\right) A_{ij}\, d\zeta - \qquad (1.6)$$

$$-\int_{-1}^{1} B_i \sum_{k=0}^{m-1} (-1)^{m+k}(2k+1)\, P_k\, d\zeta = 0 \quad (m = 1, 2, 3 \ldots)$$

Here $(B_4)_{z=0} = p_w$, whilst the remaining $(B_i)_{z=0} = 0$.

Let us use the integral relations just written down, the boundary conditions (1.3) and (1.5) at the shock wave and the boundary condition $w = 0$ at the streamlined surface, to determine the coefficients $u_m^{(N)}$, $v_m^{(N)}$, ... in Nth approximations of the required functions.

Moreover for the initial five independent equations of system (1.1) let us take the equation of continuity, the projections of the momentum equation on the axes of x and y, the equation of conservation of entropy and the equation of conservation of total entalphy in the integrated form. For a perfect gas with constant specific heats the last equation (Bernoulli's integral) has the form

$$\frac{u^2+v^2+w^2}{2} + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \frac{V^{\infty 2}}{2} + \frac{\gamma}{\gamma-1}\frac{p^{\infty}}{\rho^{\infty}} = i^{*\infty} \quad .$$

In obtaining this integral we have already used the condition at the shock wave (1.3) with $i = 6$.

Accordingly, with $V \gg 1$, for the determination of the $5(N+1)$ coefficients of the Nth approximations to the quantities $u^{(N)}$, $v^{(N)}$, $w^{(N)}$, $p^{(N)}$, $\rho^{(N)}$ and the function h , the system of relations contains the $4N$ first differential equations of the system (1.6) (with $i = 1, 2, 3, 5$ and $m = 0, 1, \ldots, N-1$), and the N final relations

$$\int_{-1}^{1} P_m\left[\rho\left(\frac{u^2+v^2+w^2}{2} - i^{*\infty}\right) + \frac{\gamma}{\gamma-1}p\right] d\zeta = 0 \quad (m = 0, 1, \ldots, N-1), \quad (1.7)$$

the four relations at the shock wave

$$B_i^{(N)} - A_{ij}^{(N)}\frac{\partial h}{\partial x_j} = B_i^{\infty} - A_{ij}^{\infty}\frac{\partial h}{\partial x_j} \qquad \text{for } \zeta = 1,\quad i = 1, 2, 3, 5$$

and one relation at the surface of the body

$$w^{(N)} = w_0^{(N)} - w_1^{(N)} + \ldots (-1)^N w_N^{(N)} = 0 \qquad \text{for } \zeta = -1 \tag{1.8}$$

With $N = 0$, i.e. in the zeroth approximation, the system of relations for determining the five quantities $u^{(0)}$, $v^{(0)}$, $w^{(0)}$, $p^{(0)}$, $\rho^{(0)}$ and h consist of the first four differential equations (1.6) with $i = 1, 2, 3, 5$, the first relation (1.7) and condition (1.8).

This system of equations of the zeroth approximation has the following form (the indices for the required quantities are dropped):

$$\begin{gathered}
\frac{\partial \rho u h}{\partial x} + \frac{\partial \rho v h}{\partial y} + \rho^\infty V_\nu = 0 \qquad \left(V_\nu = W - U\frac{\partial h}{\partial x} - V\frac{\partial h}{\partial y}\right) \\
\rho h \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial x}(p - p^\infty)h + \rho^\infty V_\nu (U - u) = 0 \\
\rho h \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}(p - p^\infty)h + \rho^\infty V_\nu (V - v) = 0 \\
\rho h \left(u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y}\right) + \rho^\infty V_\nu (S_h - S) = 0 \\
\frac{u^2 + v^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = \frac{V^{\infty 2}}{2} + \frac{\gamma}{\gamma - 1}\frac{p^\infty}{\rho^\infty}
\end{gathered} \tag{1.9}$$

Here S_h is determined by Formula (1.4).

The equation of continuity and the projections of the equation of motion on the x and y axes can also be given the following alternative form:

$$\frac{\rho h}{a^2}\left[(a^2 - u^2)\frac{\partial u}{\partial x} - uv\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + (a^2 - v^2)\frac{\partial v}{\partial y}\right] + \\
+ \rho\left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) + \frac{\rho^\infty V_\nu}{\gamma - 1}\left(\gamma\frac{S_h}{S} - 1\right) = 0$$

either $$\rho v h\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + (p - p^\infty)\frac{\partial h}{\partial x} + \rho^\infty V_\nu (U - u) - \frac{\gamma}{\gamma - 1} p h \frac{\partial \ln S}{\partial x} = 0$$

or $$\rho u h\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + (p - p^\infty)\frac{\partial h}{\partial y} + \rho^\infty V_\nu (V - v) - \frac{\gamma}{\gamma - 1} p h \frac{\partial \ln S}{\partial y} = 0$$

$$(p - p^\infty)\left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) + \rho^\infty V_\nu \left[(U - u)u + (V - v)v + \right. \\
\left. + \frac{\gamma}{\gamma - 1} p^{\frac{\gamma - 1}{\gamma}} (S_h - S)\right] = 0$$

The solution of system (1.9) can be carried out by methods analogous to those used in the solution of problems on ordinary two-dimensional gas flows (it is only necessary to assume that the equation of entropy is essentially nonlinear). In particular, for supersonic velocities we can use the method of characteristics for the solution of system (1.9).

The system (1.9) has the following families of characteristics:

$$y_{1,2}' = \frac{uv \pm a\sqrt{u^2 + v^2 - a^2}}{u^2 - a^2}, \quad y_3' = \frac{v}{u}, \quad y_4' = \frac{\Phi'_{h_y}}{\Phi'_{h_x}}$$

(the characteristics of the third family are double).

Along the characteristics the following relations obtain:

$$v' + \frac{y'}{y_1' y_2'} u' = \pm \frac{a\sqrt{u^2 + v^2 - a^2}}{v^2 - a^2} y'\Omega + \\ + \frac{a^2 y'}{\rho h (v^2 - a^2)} \left(\rho u h_x + \rho v h_y + \frac{\rho^\infty V_\nu}{\gamma - 1} \frac{\gamma S_h - S}{S} \right)$$

$$\rho h u S' + \rho^\infty V_\nu (S_h - S) = 0$$

$$\Phi_{h_x}' h_y' = -\Phi_u' \frac{\partial u}{\partial y} - \Phi_v' \frac{\partial v}{\partial y} - \Phi_\rho' \frac{\partial \rho}{\partial y} - \Phi_s' \frac{\partial S}{\partial y}$$

Here

$$\Phi \equiv (p - p^\infty)(uh_x + vh_y) + \rho^\infty V_\nu \left[(U - u) u + (V - v) v + \frac{\gamma}{\gamma - 1} p^{\frac{\gamma - 1}{\gamma}} (S_h - S) \right]$$

$$\Omega \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{1}{\rho u h} \left[(p - p^\infty) h_y + \rho^\infty V_\nu (V - v) - \frac{\gamma}{\gamma - 1} ph \frac{\partial \ln S}{\partial y} \right]$$

The quantities $\partial u / \partial y$, $\partial v / \partial y$, $\partial \rho / \partial y$, $\partial S / \partial y$ are easily expressed in terms of the corresponding values of u', v', ρ' and S', for example

$$\rho h (v - uy') \frac{\partial S}{\partial y} = -\rho h u S' - \rho^\infty V_\nu (S_h - S)$$

The first three families of characteristics are the usual acoustic characteristics and streamlines of two-dimensional problems of gas dynamics. The fourth family is a new one, having no analog in the ordinary two-dimensional problems of gas dynamics.

To solve the system thus obtained we need to formulate the boundary conditions on the boundary of the region in the xy-plane. In what follows we shall restrict ourselves to the case of flow past a plane wing with sharp edges. Then the boundary of the flow region under consideration is the edge of the wing.

It is evident that on the part of the contour where the shock wave is attached to the edge of the wing, $h = 0$, whilst the values of the remaining required functions are determined from the relations on the wave.

The boundary conditions on the remaining part of the contour in the general case cannot be specified in advance, so that the flow on the pressure side of the wing and the flow on its suction side have to be calculated together.

Let us suppose ideally that with fixed conditions in the free stream the pressure on the suction side of the wing is reduced. The influence of this decrease of pressure will be transmitted to the pressure side of the wing along that part of the edge where the shock wave is detached and the velocity component of the gas normal to the edge is less than the sound velocity. As the pressure is lowered this component will grow until it reaches the sound velocity; after that the influence of a fall in the pressure on the flow on the pressure side of the wing will cease.

Accordingly, for a sufficiently large velocity of the free stream and large angles of attack, when the ratio of the pressures on the pressure and suction sides of the wing is sufficiently large, we must take $v_n \geqslant a$ for the boundary condition at the boundary of the region (i.e. at the edge of the wing), where $h \neq 0$.

2. The linearized equations and their solution. Suppose that the leading edge of the wing has a straight line segment, and moreover that on this section the shock wave is attached to the edge and that the flow behind it is supersonic. Then in the region of influence of the straight segment of the edge the system of Equations (1.9) gives an exact solution, corresponding to a translational flow of gas. If the segment of the edge differs only slightly from a straight line, then to find the flow in the region of influence, and also in a certain neighborhood outside it, we can make use of the linearized equations.

Let us assume that the difference in the stream behind the shock from a translational stream is characterized by the small parameter ε. For example, let us assume that the equation of the leading edge of the wing at the segment under consideration has the form $x = x^*(y)$, where $x^*(y) = \varepsilon x_1(y)$, and x is a quantity of the order of unity. Let us write the solution of the system (1.9) in the form of series in ε

$$u = u_0 + \varepsilon u_1 + \ldots, \quad v = v_0 + \varepsilon v_1 + \ldots, \quad p = p_0 + \varepsilon p_1 + \ldots,$$
$$\rho = \rho_0 + \varepsilon\rho_1 + \ldots, \quad S = S_0 + \varepsilon S_1 + \ldots, \quad h = k(x - x^*) + \varepsilon h_1 + \ldots \qquad (2.1)$$

and let us restrict ourselves in what follows to the determination only of the terms written down in these series. Let us substitute Expressions (2.1) and also Expression $S_h = S_{h0} + \varepsilon S_{h1} + \ldots$ in Equation (1.9). Bearing in mind that in the case under consideration the quantities with the subscript 0 and the quantity k are constants, we obtain for the determination of these quantities the following system of relations (the relations at the shock):

$$\rho_0 u_0 k + \rho^\infty (W - Uk) = 0, \quad p_0 - p^\infty - \rho_0 u_0 (U - u_0) = 0, \quad v_0 = V \qquad (2.2)$$
$$p_0 - p^\infty - \rho^\infty W (W - Uk) = 0, \quad \frac{u_0^2 + v_0^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p_0}{\rho_0} = \frac{V_\infty^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p^\infty}{\rho^\infty}$$

Here instead of the equation $S_{h0} = S_0$, obtained by using the momentum equation projected on the normal to the wing, we have written down this equation itself. The system for the determination of the following terms of the series (2.1) has the form

$$\frac{\partial}{\partial x}[\rho_0 k (x - x^*) u_1 + u_0 k (x - x^*) \rho_1] + \frac{\partial}{\partial y}[\rho_0 k (x - x^*) v_1 + v_0 k (x - x^*) \rho_1] +$$
$$+ (\rho_0 u_0 - \rho^\infty U)\frac{\partial h_1}{\partial x} + (\rho_0 v_0 - \rho^\infty V)\left(\frac{\partial h_1}{\partial y} - k x_1'\right) = 0$$

$$\frac{\partial}{\partial x}[2\rho_0 u_0 k (x - x^*) u_1 + u_0^2 k (x - x^*) \rho_1 + k (x - x^*) p_1] + \qquad (2.3)$$
$$+ \frac{\partial}{\partial y}[\rho_0 u_0 k (x - x^*) v_1 + \rho_0 v_0 k (x - x^*) u_1 + u_0 v_0 k (x - x^*) \rho_1] +$$
$$+ U (\rho_0 u_0 - \rho^\infty U)\frac{\partial h_1}{\partial x} + (\rho_0 u_0 v_0 - \rho^\infty U V)\left(\frac{\partial h_1}{\partial y} - k x_1'\right) = 0$$

(to be continued)

$$\frac{\partial}{\partial x}[\rho_0 u_0 k(x-x^*)v_1+\rho_0 v_0 k(x-x^*)u_1+u_0 v_0 k(x-x^*)\rho_1]+$$
$$+\frac{\partial}{\partial y}[2\rho_0 v_0 k(x-x^*)v_1+v_0^2 k(x-x^*)\rho_1+k(x-x^*)p_1]+$$
$$+(\rho_0 u_0-\rho^\infty U)V\frac{\partial h_1}{\partial x}+(\rho_0 v_0^2+p_0-p^\infty-\rho^\infty V^2)\left(\frac{\partial h_1}{\partial y}-kx_1'\right)=0$$
$$\frac{\partial}{\partial x}u_0(x-x^*)S_1+\frac{\partial}{\partial y}v_0(x-x^*)S_1-u_0 S_{h_1}=0$$
$$u_0 u_1+v_0 v_1+\frac{\gamma}{\gamma-1}\left(\frac{p_1}{\rho_0}-\frac{p_0}{\rho_0^2}\rho_1\right)=0$$
$$\frac{S_1}{S_0}=\frac{1}{\gamma}\frac{p_1}{p_0}-\frac{\rho_1}{\rho_0},\qquad S_{h_1}=m\frac{\partial h_1}{\partial x}+n\left(\frac{\partial h_1}{\partial y}-kx_1'\right)$$
$$m=-\frac{2(W-Uk)(U+Wk)}{(1+k^2)^2}S_h',\qquad n=-\frac{2(W-Uk)V}{1+k^2}S_h'$$

Here S_h' denotes the derivative of S_h with respect to v_n^2 when $\epsilon = 0$, i.e.

$$S_h'=\frac{2(\gamma-1)}{\gamma(\gamma+1)}\frac{(1-a^{\infty 2}/v_n^2)^2}{p_0^{\gamma-1/\gamma}}\qquad\left(v_n^2=\frac{(W-U\,\partial h/\partial x-V\,\partial h/\partial y)^2}{1+(\partial h/\partial x)^2+(\partial h/\partial y)^2}\right)$$

By the substitution

$$\begin{aligned}&k(x-x^*)u_1=U_\times, && k(x-x^*)p_1=P_\times, && k(x-x^*)S_1=\sigma_\times\\ &k(x-x^*)v_1=V_\times, && k(x-x^*)\rho_1=R_\times, && h_1-kx_1=H_\times\end{aligned}\qquad(2.4)$$

the system of linear equations (2.3) is reduced to a system of linear equations with constant coefficients

$$\frac{\partial}{\partial x}(\rho_0 U_\times+u_0 R_\times)+\frac{\partial}{\partial y}(\rho_0 V_\times+v_0 R_\times)+$$
$$+(\rho_0 u_0-\rho^\infty U)\frac{\partial H_\times}{\partial x}+(\rho_0 v_0-\rho^\infty V)\frac{\partial H_\times}{\partial y}=0$$
$$\frac{\partial}{\partial x}(\rho_0 u_0 U_\times+P_\times)+\frac{\partial}{\partial y}\rho_0 v_0 U_\times+$$
$$+(U-u_0)(\rho_0 u_0-\rho^\infty U)\frac{\partial H_\times}{\partial x}-\rho^\infty V(U-u_0)\frac{\partial H_\times}{\partial y}=0$$
$$\frac{\partial}{\partial x}\rho_0 u_0 V_\times+\frac{\partial}{\partial y}(\rho_0 v_0 V_\times+P_\times)+\rho_0 u_0(U-u_0)\frac{\partial H_\times}{\partial y}=0$$
$$\frac{\partial}{\partial x}u_0\sigma_\times+\frac{\partial}{\partial y}v_0\sigma_\times-u_0 k\left(m\frac{\partial H_\times}{\partial x}+n\frac{\partial H_\times}{\partial y}\right)=0$$
$$u_0 U_\times+v_0 V_\times+\frac{\gamma}{\gamma-1}\left(\frac{P_\times}{\rho_0}-\frac{p_0}{\rho_0^2}R_\times\right)=0,\qquad\frac{\sigma_\times}{S_0}=\frac{1}{\gamma}\frac{P_\times}{p_0}-\frac{R_\times}{\rho_0}$$

This system has four real characteristic directions, determined by the relations $dy/dx = \beta_i$. Corresponding to this direction the solutions have the form

$$\beta_1 = \frac{u_0 v_0 + a_0 \sqrt{a_0^2 - (u_0^2 + v_0^2)}}{u_0^2 - a_0^2}, \qquad \beta_2 = \frac{u_0 v_0 - a_0 \sqrt{a_0^2 - (u_0^2 + v_0^2)}}{u_0^2 - a_0^2}, \qquad \beta_3 = \frac{v_0}{u_0}$$

$$U_{\times 1} = -\beta_1 V_\times^{(1)}, \quad V_{\times 1} = V_\times^{(1)}(y - \beta_1 x), \quad U_{\times 2} = -\beta_2 V_\times^{(2)}, \quad V_{\times 2} = V_\times^{(2)}(y - \beta_2 x)$$

$$P_1 = \rho_0(\beta_1 u_0 - v_0) V_\times^{(1)}, \quad \sigma_{\times 1} = 0, \qquad P_{\times 2} = \rho_0(\beta_2 u_0 - v_0) V_\times^{(2)}, \quad \sigma_{\times 2} = 0$$

$$R_{\times 1} = \frac{\rho_0}{a_0^2}(\beta_1 u_0 - v_0) V_\times^{(1)}, \quad H_{\times 1} = 0, \qquad R_{\times 2} = \frac{\rho_0}{a_0^2}(\beta_2 u_0 - v_0) V_\times^{(2)}, \quad H_{\times 2} = 0$$

$$U_{\times 3} = -\frac{u_0 a_0^2}{(\gamma - 1) S_0 (u_0^2 + v_0^2)} \sigma_\times, \qquad P_{\times 3} = 0, R_{\times 3} = -\frac{\rho_0}{S_0} \sigma_\times$$

$$V_{\times 3} = -\frac{v_0 a_0^2}{(\gamma - 1) S_0 (u_0^2 + v_0^2)} \sigma_\times, \quad \sigma_{\times 3} = \sigma_\times (y - \beta_3 x), \qquad H_{\times 3} = 0$$

When $\beta_4 = (1 + k^2) v_0 / u_0 \neq \beta_3$ (i.e. when $V \neq 0$)

$$U_{\times 4} = (U - u_0 - \beta_4 C) H_\times, \qquad R_{\times 4} = \frac{\rho_0}{a_0^2}[(\beta_4 u_0 - v_0) C - u_0 (U - u_0)] H_\times$$

$$V_{\times 4} = \frac{(U - u_0)(\beta_4 a_0^2 - k^2 u_0 v_0)}{a_0^2 (1 + \beta_4^2) - k^4 v_0^2} H_\times = C H_\times, \qquad \sigma_{\times 4} = 0$$

$$P_{\times 4} = \rho_0 [(\beta_4 u_0 - v_0) C - u_0 (U - u_0)] H_\times, \qquad H_{\times 4} = H_\times (y - \beta_4 x)$$

If $V = 0$, then $\beta_3 = \beta_4 = 0$, and the functions corresponding to this common characteristic direction have the form

$$U_\times = (U - u_0) H_\times - \frac{a_0^2}{(\gamma - 1) u_0 S_0} \sigma_\times, \qquad R_\times = -\frac{p_0 - p^\infty}{a_0^2} H_\times - \frac{\rho_0}{S_0} \sigma_\times$$

$$V_\times = 0, \qquad P_\times = -(p_0 - p^\infty) H_\times, \qquad \sigma_\times = \sigma_\times(y), \qquad H_\times = H_\times(y)$$

The solution obtained shows that, as in ordinary problems of plane supersonic gas flows, only perturbations of entropy, density and longitudinal velocity are transmitted along the streamlines of the unperturbed motion (it is easily seen that in the system of coordinates in which $v_0 = 0$ the equation $V_{\times 3} = 0$ holds). Perturbations of pressure, density and the component of velocity perpendicular to the characteristic are transmitted along the acoustic characteristics. The characteristics of the fourth family are lines of transmission of perturbations in the form of the shock wave (the thickness of the layer of compressed gas); perturbations of enropy are not transmitted along these characteristics.

A straightforward geometrical consideration shows that the fourth characteristic direction is the direction of the projection on the plane of the wing of the velocity component of the free stream tangential to the shock wave. More obvious is another interpretation of this direction. In the flow behind the shock wave let us consider the Mach cone issuing from a point of the edge of the wing. This cone intersects the plane of the shock along two straight lines. We can show that the fourth characteristic direction is the bisector of the angle between those lines which are the projections of these two lines on the plane of the wing. Certainly, according to the physical meaning established above cf the characteristics of the fourth family, as lines of propagation of perturbations in the form of the wave, it would be more satisfactory if each of these straight lines separately were a characteristic. However, in the zeroth approximation we do not succeed in obtaining this result.

The four arbitrary functions $V_\times^{(1)}(y-\beta_1 x)$, $V_\times^{(2)}(y-\beta_2 x)$, $\sigma_\times(y-\beta_3 x)$, $H_\times(y-\beta_4 x)$, appearing in the general solution, are easily determined in the region of influence by means of the equation of the leading edge $x_1(y)$.

Indeed, in accordance with the definition (2.4), the functions $U_\times$, $V_\times$, $P_\times$, $R_\times$, $\sigma_\times$ must vanish when $x = x^*$ (the quantities u_1, v_1, p_1, ρ_1, S_1 remain bounded when approaching the edge of the wing), whilst the function $H_\times$ is determined from the relation

$$H_\times(y-\beta_4 x^*) = -kx_1(y) \tag{2.5}$$

Representing each of the functions $U_\times, V_\times, P_\times, R_\times, \sigma_\times$ in the form of a sum of arbitrary functions $V_\times^{(1)}$, $V_\times^{(2)}$, $\sigma_\times$, $H_\times$ with corresponding coefficients and equating to zero when $x = x^*$, we obtain

$$U_\times = -\beta_1 V_\times^{(1)} - \beta_2 V_\times^{(2)} - \frac{u_0 a_0^2}{(\gamma-1)S_0(u_0^2+v_0^2)}\sigma_\times + (U-u_0-\beta_4 C)H_\times = 0$$

$$V_\times = V_\times^{(1)} + V_\times^{(2)} - \frac{v_0 a_0^2}{(\gamma-1)S_0(u_0^2+v_0^2)}\sigma_\times + CH_\times = 0$$

$$P_\times = \rho_0(\beta_1 u_0 - v_0)V_\times^{(1)} + \rho_0(\beta_2 u_0 - v_0)V_\times^{(2)} + \rho_0[(\beta_4 u_0 - v_0)C - u_0(U-u_0)]H_\times = 0$$

$$R_\times = \frac{\rho_0}{a_0^2}(\beta_1 u_0 - v_0)V_\times^{(1)} + \frac{\rho_0}{a_0^2}(\beta_2 u_0 - v_0)V_\times^{(2)} - \frac{\rho_0}{S_0}\sigma_\times + \frac{\rho_0}{a_0^2}[(\beta_4 u_0 - v_0)C - u_0(U-u_0)]H_\times = 0$$

$$\sigma_\times = 0$$

Bearing in mind the last equation, we find that the two previous equations are consequences of the first two (in the absence of any entropy perturbation it follows from the vanishing of perturbations in the velocity components that there are no perturbations in the pressure and density). From the first two equations with $\sigma_\times = 0$ we find that

$$V_\times^{(1)}(y-\beta_1 x^*) = -\frac{(\beta_2-\beta_4)C + U - u_0}{\beta_1-\beta_2}kx_1(y)$$

$$V_\times^{(2)}(y-\beta_2 x^*) = -\frac{(\beta_1-\beta_4)C + U - u_0}{\beta_2-\beta_1}kx_1(y)$$

In the particular case when $V = 0$, we find that

$$C = 0,\ \beta_1 = -\beta_2 = (M_0^2-1)^{-1/2}$$

and hence

$$V_\times^{(1)}(y-\beta_1 x^*) = -V_\times^{(2)}(y+\beta_1 x^*) = \frac{U-u_0}{2\beta_1}kx_1(y) \tag{2.6}$$

Let us consider examples.

Let the leading edge of the wing consist of two straight line segments, the equations of which are

$$x = 0 \text{ when } y < 0, \quad x = \varepsilon y \text{ when } y > 0$$

i.e. let the function $x_1(y)$ in Expressions (2.5) and (2.6) be defined by Formulas

$$x_1(y) = 0 \quad \text{for } y < 0, \qquad x_1(y) = y \quad \text{for } y > 0$$

Then

$$V_\times^{(1)}(\xi) = -\frac{k}{2\beta}(U - u_0)(1 + \beta\varepsilon)\xi \quad \text{for } \xi > 0$$

$$V_\times^{(2)}(\eta) = \frac{k}{2\beta}(U - u_0)(1 - \beta\varepsilon)\eta \quad \text{for } \eta > 0$$

$$H_\times(y) = -ky \quad \text{for } y > 0$$

For negative values of the arguments all these functions are equal to zero.

Let us find expressions for the pressure and for the velocity components v, defining in the linear approximation the form of the streamlines, in the four regions separated from one another by the characteristics

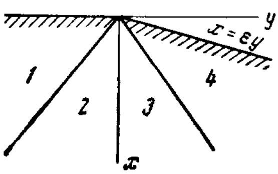

Fig. 1

$$\eta \equiv y + \beta x = 0, \quad y = 0, \quad \xi \equiv y - \beta x = 0 \text{ (see Fig.1)}$$

Regions	p_1	v_1
1	0	0
2	$-\frac{1}{2}(p_0 - p^\infty)((y/x) + \beta)$	$\frac{U - u_0}{2\beta}\left(\frac{y}{x} + \beta\right)$ (regions 2 and 3)
3	$\frac{1}{2}(p_0 - p^\infty)((y/x) - \beta)$	
4	0	$U - u_0$

Inside the angle formed by the characteristics $y + \beta x = 0$ and $y - \beta x = 0$ the pressure is reduced, and the greatest reduction of pressure is equal to

$$-\varepsilon \frac{p_0 - p^\infty}{2\sqrt{M_0^2 - 1}} \qquad \text{for } y = 0$$

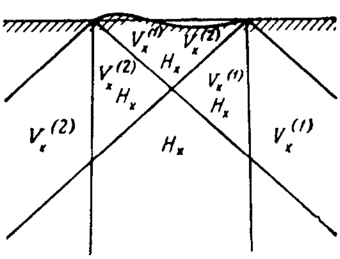

Fig. 2

The variation of pressure in region 4, in comparison with the pressure in region 1, is of order ε^2. All the streamlines are inclined to the side of positive y, asymptotically assuming the direction of the line $y = \varepsilon((U - u_0)/2u_0)x$ (this direction corresponds to the singular point of Ferri type for conical three-dimensional flows), bisecting the angle between the directions of the translational flows in regions 1 and 4.

It is evident that the solution of the problem for the case when $x_1 = |y|$, i.e. for symmetrical flow past a wing, can be obtained in the linear formulation by a simple superposition of the solution just found.

For the next example let us consider (Fig.2) a wing for which the leading edge is a straight line perpendicular to the free stream everywhere except for the segment $(-l, l)$. In each of the regions separated from one another by the characteristics issuing from the ends of the curved segment of the edge, the functions determining the perturbation of the stream are shown in Fig.2. In determining these functions by Formulas (2.5) and (2.6) and using them in the region of influence of the curved segment of the edge it is necessary to retain the terms of order ε. In Expressions (2.4), defining the perturbation of the flow by the functions $V_\times^{(1)}, V_\times^{(2)}$ and $H_\times$; the term $x^* = \varepsilon x_1$ must not be neglected in comparison with x. Accordingly, in the region of influence of the curved segment of the edge we obtain

$$[x - \varepsilon x_1(y)]\, v_1 = \frac{U - u_0}{2\beta} \{x_1[y + \beta(x - \varepsilon x_1)] - x_1[y - \beta(x - \varepsilon x_1)]\}$$

$$v_1 \to (U - u_0)\, x_1'(y) \qquad \text{for} \quad x \to \varepsilon x_1(y)$$

i.e. v_1 tends to the required value determined by the relations at the shock. Similarly,

$$(x - x^*)\, p_1 = (p_0 - p^\infty)\{x_1(y) - {}^1/_2(x_1[y - \beta(x - x^*)] + x_1[y + \beta(x - x^*)]\}$$

For small values of $x - x^*$

$$p_1 \to {}^1/_2(p_0 - p^\infty)\, \beta^2 x_1''(y)(x - x^*)$$

At very remote points in the region of influence of the curved portion

$$p_1 = \frac{p_0 - p^\infty}{\sqrt{M_0^2 - 1}} \frac{x_1(0)}{l}$$

Let us consider now a wing having the plan form of an isosceles triangle with the base turned towards the free stream. If the shock is attached to the leading edge then so long as the whole of the wing lies in the region of influence of the leadind edge, the flow on the surface of the wing is translational. If we increase the angle of attack (or decrease the Mach number M of the free stream, or lengthen the wing, decreasing the angle opposite the leading edge), then between the region of influence of the leading edge of the wing and the lateral edges of the wing there are formed regions of flow with veritable parameters (Fig.3). To make possible the use of the linear theory we shall suppose that the equation of one of the edges has the form

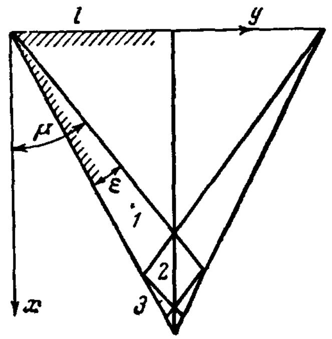

Fig. 3

$$y = x \tan(\mu - \varepsilon)$$

where $\tan\mu = \beta$, and ε is a small quantity. At the edge of the wing we must have the condition $v_n = a$, which in the linear approximation can be reduced to the following form:

$$V_\times^{(1)} + AV_\times^{(2)} = -\frac{2}{\gamma + 1} k u_0 x \cos^2\mu \qquad \text{for} \quad y \approx x\left(\beta - \frac{\varepsilon}{\cos^2\mu}\right)$$

$$A = \frac{2}{\gamma + 1}\left(\frac{3 - \gamma}{2} - \frac{2}{M_0^2}\right)$$

On the line of symmetry of the wing, i.e. $y = l$, it follows from the condition $v = 0$ that

$$V_\times^{(1)} + V_\times^{(2)} = 0$$

From these conditions it is easy to find that in region 1

$$V_\times^{(1)} = \frac{2}{\gamma + 1} \frac{k u_0 \cos^4\mu}{\varepsilon} \xi, \qquad V_\times^{(2)} = 0, \qquad H_\times = 0$$

In the region 2 the quantities $V_\times^{(1)}$ and $H_\times$, obviously remain the same, whilst

$$V_\times^{(2)} = \frac{2}{\gamma + 1} \frac{k u_0 \cos^4\mu}{\varepsilon} (\eta - 2l)$$

The solution in regions 3, 4 and so on, can also be found without difficulty, but, bearing in mind the size of these regions, in the linear approximation it is sufficient to limit consideration of the flow just to regions 1 and 2.

Accordingly, in region 1, where the flow is conical in character,

$$v = \frac{2}{\gamma+1} u_0 \cos^4\mu \left(\frac{y}{x} - \beta\right), \qquad p = p_0 + \frac{2}{\gamma+1} \rho_0 u_0^2 \tan\mu \cos^4\mu \left(\frac{y}{x} - \beta\right)$$

In region 2

$$v = \frac{4}{\gamma+1} u_0 \tan\mu \cos^4\mu \left(\frac{y}{l} - 1\right), \qquad p = p_0 - \frac{4}{\gamma+1} \rho_0 u_0^2 \sin^2\mu \cos^2\mu \left(\frac{x\beta}{l} - 1\right)$$

At the point with the greatest value of x in region 2

$$p = p_0 - \frac{4\varepsilon}{\gamma+1} \rho_0 u_0^2 \sin\mu \cos\mu$$

The coefficient of normal force acting on the wing has the form

$$C_N - C_N^\circ = \begin{cases} 0 & \text{for } \mu < \theta_0 \\ -\dfrac{2}{\gamma+1} \dfrac{\rho_0 u_0^2}{\rho^\infty V_\infty^2} \sin\theta_0 \cos\theta_0 (\mu - \theta_0)^2 & \text{for } \mu > \theta_0 \end{cases}$$

Here C_N° is the value of C_N for the wing of infinite span.

3. Conical flows. As an example of the use of the nonlinear equations let us consider conical flows. These flows describe certain behaviors of flow past triangular, trapezoidal and other wings, the leading edge of which consists of straight line segments. In conical flows the parameters of the gas in the layer between the surface of the wing and the shock wave do not depend on the distance r from the vertex of the wing (taken as the origin of coordinates), but the thickness of the layer h is proportional to this distance. Measuring the polar angle θ from the direction of the vector velocity of the free stream on the plane of the wing, denoting by u and v the radial and circumferential velocities, and setting $h = rH(\theta)$, let us transform Equation (1.9), taking account of the conical nature of the flow, into the form

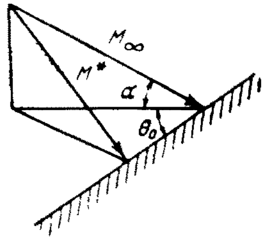

Fig. 4

$$2\rho u H + \frac{d}{d\theta} \rho v H + \rho^\infty V_\nu = 0 \qquad (3.1)$$

$$\rho v H \frac{du}{d\theta} - \rho v^2 H + (p - p^\infty) H + \rho^\infty V_\nu (V_r - u) = 0$$

$$(p - p^\infty)\left(uH + v\frac{dH}{d\theta}\right) + \rho^\infty V_\nu \Big[u(V_r - u) +$$

$$+ v(V_\theta - v) + \frac{\gamma}{\gamma-1} p^{\gamma-1/\gamma}(S_h - S)\Big] = 0$$

$$\rho v H \frac{dS}{d\theta} + \rho^\infty V_\nu (S_h - S) = 0$$

$$\left(V_\nu = W - V_r H - V_\theta \frac{dH}{d\theta}, \; W = -V_\infty \sin\alpha, \; V_r = V_\infty \cos\alpha\cos\theta, \; V_\theta = -V_\infty \cos\alpha \sin\theta\right)$$

The quantity $S_h(v_n^2)$ is determined by Expression (1.4), where

$$v_n^2 = \frac{(W - V_r H - V_\theta dH / d\theta)^2}{1 + H^2 + (dH / d\theta)^2}$$

The third equation of system (3.1) determines $dH/d\theta$ implicitly. The remaining equations can be solved with respect to the derivatives, as a result of which we obtain

$$\Phi\,(dH\,/\,d\theta,\ H,\ u,\ v,\ S) = 0 \tag{3.2}$$

$$\frac{du}{d\theta} = v - \frac{(p - p^{\infty})\,H + \rho^{\infty}\,(W - V_r H - V_\theta dH\,/\,d\theta)\,(V_r - u)}{\rho v H}$$

$$\frac{dv}{d\theta} = \frac{1}{\rho H\,(v^2 - a^2)}\Big[\rho a^2(2uH + vH') - \rho u v^2 H + (p - p^{\infty})uH + \rho^{\infty}V_\nu\Big\{a^2\Big[\,1 + \frac{S_h - S}{S}\,\frac{\gamma}{\gamma - 1}\Big] + u\big(V_r - u\big)\Big\}\Big]$$

$$\frac{dS}{d\theta} = -\,\frac{\rho^{\infty}V_\nu\,(S_h - S)}{\rho v H}, \qquad \frac{u^2 + v^2}{2} + \frac{\gamma}{\gamma - 1}\,S p^{\gamma - 1/\gamma} = i^{*\infty}, \qquad \rho = \frac{p^{1/\gamma}}{S}, \qquad a^2 = \gamma\,\frac{p}{\rho}$$

Let us consider the portion of the edge of the wing characterized by the angle $\theta_0 < \frac{1}{2}\pi$. Simple geometrical consideration (Fig.4) shows that if the angle θ_0 is such that the following condition is fulfilled:

$$\sin\theta_0\,/\tan\alpha > \cot\varphi_{max}\,(M^*) \qquad (M^* = \sqrt{M_\infty^2\sin^2\alpha + M_\infty^2\cos^2\alpha\,\sin^2\theta_0}) \tag{3.3}$$

where $\varphi_{max}\,(M)$ is the limiting angle of deflection of the stream in an oblique shock wave for a given number M, then the shock is attached along the edge. The inequality (3.3) can be rewritten in the form

$$\Omega > \cot\varphi_{max}\,(K\sqrt{1 + \Omega^2}) \qquad (K = M_\infty \sin\alpha,\ \Omega = \sin\theta_0\,/\tan\alpha) \tag{3.4}$$

The region corresponding to the fulfilment of this inequality has values of the parameters M_∞, α and θ_0 lying (with $\gamma = 1.4$) above the curve depicted in Fig.5. The boundary values of the required functions at the edge of the wing are in this case determined by the relations at the shock and the condition $H = 0$. For values of the parameters M_∞, α and θ_0 not satisfying the inequality (3.4), the shock in the case where conical flow exists is attached only at the vertex of the angle of the leading edge. Then the gas from the pressure side of the wing flows round the edge to the suction side. For sufficiently large values of the pressure ratio between the pressure and suction sides of the wing the component of velocity normal to the edge must reach sonic velocity at the edge, i.e. $v = a$ when $\theta = \theta_0$.

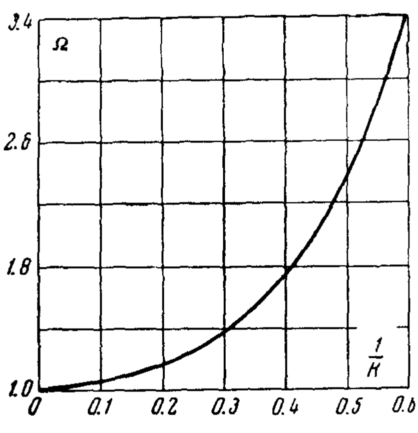

Fig. 5

This equation, or the relation $v' = \infty$ when $\theta = \theta_0$ which is equivalent to it, following from Expression (3.2) for v', constitutes the boundary condition in the solution of the system (3.2) in the case when the defining parameters M_∞, α and θ_0 belong to the region below the curve in Fig.5.

Let us consider two examples. Suppose that the leading edge of the wing forms an angle, one side of which is perpendicular to the direction of the free stream (i.e. $\theta = -\frac{1}{2}\pi$ there), whilst the other is characterized by the angle $-\frac{1}{2}\pi < \theta_0 < \frac{1}{2}\pi$. The angle of attack of the wing α will be assumed to be such that the shock for $\theta = -\frac{1}{2}\pi$ would be attached to the edge. It is evident that in such case for $\theta_0 = \frac{1}{2}\pi$ there is a simple exact solution in which

$$u = u_0 \cos\theta, \qquad v = -u_0 \sin\theta$$
$$H = k\cos\theta$$

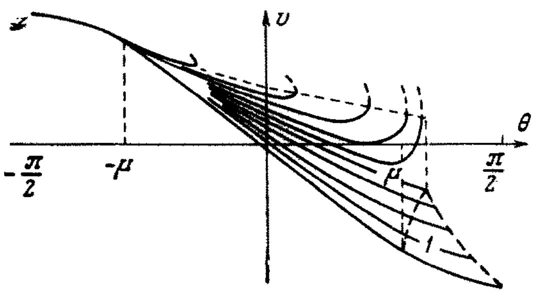

Fig. 6

whilst the quantities p and ρ are constants (translational flow behind the shock); u_0, k, p and ρ are found from the relations (2.2) at the shock (when $V = 0$).

We shall begin by gradually decreasing the angle θ_0 . The flow resulting from this can be divided into the three following regions: two translational flows with $-\frac{1}{2}\pi < \theta < -\mu$ and with $\mu_0 < \theta < \theta_0$ and included between them a flow with variable parameters. We denote by $-\mu$ and μ_0 the angles formed by the bounding characteristics of both translational flows with the direction $\theta = 0$. The parameters of the gas in both translational flows and, in particular, these angles are determined with the help of the shock relations. To describe the flows arising with variation of θ_0, we turn to Figs 6 and 7, in which are depicted, respectively, the quantities v and $a^2 - v^2$ as functions of θ for fixed M and α and various angles θ_0.

When $\theta_0 = \frac{1}{2}\pi$ we have $v = -u_0 \sin\theta$ (the lowest curve in Fig.6) and $a^2 - v^2 = u_0^2(\sin^2\mu - \sin^2\theta)$ (the uppermost curve in Fig.7). As θ_0 decreases the solution in the interval $\mu_0 < \theta < \theta_0$ is easily found from the relations at the shock. The corresponding curves are sketched in Figs. 6 and 7 region 1. For construction of the curves in the interval $-\mu < \theta < \mu_0$ (in the case when the shock is attached), or in the intrval $-\mu < \theta < \theta_0$ (in the case when the shock becomes detached) we notice that the point $\theta = -\mu$ (corresponding to the characteristic) is singular for the equation determining $dv/d\theta$. Indeed, this equation can be rewritten in the form

$$\frac{dv}{d\theta} = -u - \frac{1}{\rho H(a^2 - v^2)}\left\{u\left[(p - p^\infty)H + \rho^\infty V_\nu (V_r - u)\right] + a^2\left[\rho(uH + vH') + \rho^\infty V_\nu\right] + \right.$$
$$\left. + \frac{\gamma}{\gamma - 1}\rho^\infty V_\nu a^2\left(\frac{S_h}{S} - 1\right)\right\}$$

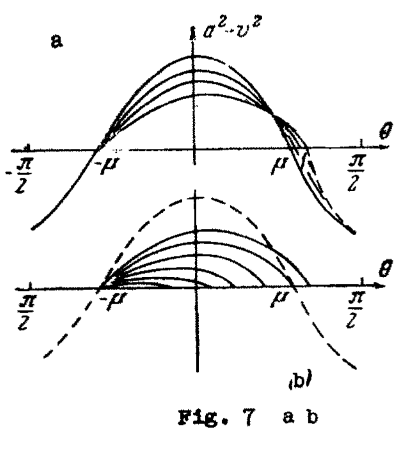

Fig. 7 a b

The numerator and denominator of the second term in the right-hand side evidently vanish when $\theta = -\mu$ (of course the same applies when $\theta = \mu_0$). Accordingly, from the point $\theta = -\mu$ there issues a pencil of curves $v(\theta)$, which differ in the initial value of the derivative $dv/d\theta$ and, consequently, a pencil of curves $a^2 - v^2$. If we introduce the notation

$$\left.\frac{dv}{d\theta}\right|_{\theta=-\mu} = -u_0 + \Delta$$

then it is easy to show that

$$\left.\frac{d}{d\theta}(a^2 - v^2)\right|_{\theta=-\mu} = [2u_0 - (\gamma + 1)\Delta]\,v_0$$

As Δ increases from zero the curves of $v(\theta)$ and $a^2 - v^2$ extend up to $\theta = \mu_0$, where they join up (with a discontinuity of derivatives) with the corresponding segments in the region 1, relating to translational flow. Starting with a certain value Δ, the vanishing of the difference $a^2 - v^2$ occurs earlier than the vanishing of the numerator of the second term in the expression for $dv/d\theta$; the derivative $dv/d\theta$ becomes infinite at such a point. It is obvious that the solutions for such Δ correspond to flows with detached shocks, whilst the value of θ, at which $a^2 - v^2$ vanishes and v' is infinite, is just the angle θ_0 of the edge of the wing. The behavior of the curves described is illustrated in Figs. 6 and 7. The expressions derived above for $dv/d\theta$ and $d(a^2 - v^2)/d\theta$ when $\theta = -\mu$ show that the initial values of the derivative $dv/d\theta$ are included between the limits

$$-u_0 \leqslant \left.\frac{dv}{d\theta}\right|_{\theta=-\mu} \leqslant -\frac{\gamma-1}{\gamma+1}u_0$$

Using Figs. 6 and 7, it is easy to construct the pattern of the streamlines and the acoustic characteristics for the qualitatively distinct cases of flow past a wing. In Figs. 8 a, b and c is shown the successive replacement of the behaviors of flow as the angle θ_0 is decreased. The streamlines are shown as full lines, and the two families of acoustic characteristics as broken lines. Fig. 8 a corresponds to the behaviors of flow with an attached shock wave. In the region of conical flow with variable parameters there is one straight streamline ($v = 0$), the direction of which is asymptotic for all the other streamlines. Such a peculiarity of behavior in conical flows is well known; as already noticed above, in three-dimensional conical flows the presence of the corresponding singular points was established by A.Ferri. Fig.8, b relates to flows with detached shock, for which the curve of $v(\theta)$ has a segment with negative values of v (Fig.6). In this case there occurs one more straight streamline. From this second streamline the flow diverges outside across the edge of the wing, and inside asymptotically approaching the direction of the first straight streamline. On further decrease of the angle θ_0 both straight streamlines merge and disappear; all the streamlines then intersect the edge of the wing, as shown in Fig.8 c. Further decrease of the angle θ_0, failing to change the qualitative pattern of the flow, leads to contraction of the region with streamlines directed outwards. Finally, when $\theta_0 \leqslant -\mu$ the whole surface of the wing is occupaied by a translational stream of gas.

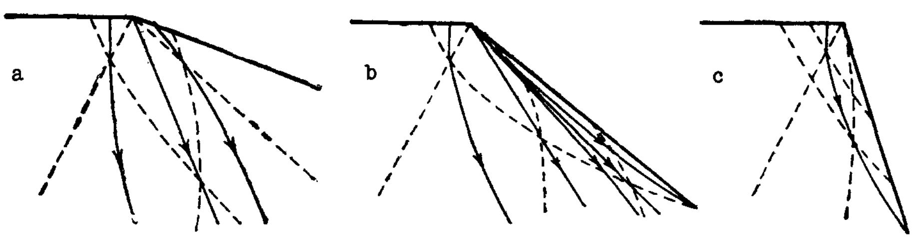

Fig. 8 a, b, c

For the second example let us consider symmertical flow past an angle, i.e. let us take $\theta_1 = -\theta_0$ and let us assume, moreover, that $\theta_0 \leqslant {}^1/_2\pi$. Obviously it is sufficient to consider the solution only for $0 \leqslant \theta \leqslant \theta_0$.

On the line of symmetry $\theta = 0$ the conditions $v = 0$, $dH/d\theta = 0$, $dS/d\theta = 0$ must be satisfied. The vanishing of the derivative $du/d\theta$ follows from these conditions and the alternative form of the expression for $du/d\theta$

$$\frac{du}{d\theta} = v + \frac{1}{\rho u H}\left[\left(p - p^{\infty}\right)\frac{dH}{d\theta} + \rho^{\infty} V_{\nu}(V_{\theta} - v) - \frac{\gamma}{\gamma-1}\frac{pH}{S}\frac{dS}{d\theta}\right]$$

The system of equations (3.1) gives the following connection between the values of the required functions when $\theta = 0$

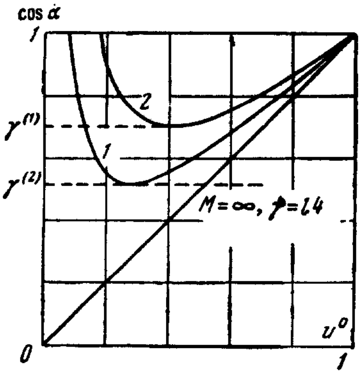

Fig. 9

$$(p_0 - p^{\infty}) H_0 + \rho^{\infty} (W - V_{r0} H_0)(V_{r0} - u_0) = 0 \tag{3.5}$$

$$S_{h0}(v_{n0}^2) = S_0, \qquad v_{n0}^2 = \frac{(W - V_{r0}H_0)^2}{1 + H_0^2}$$

$$\frac{u_0^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p_0}{\rho_0} = i^{*\infty}, \qquad S_0 = \frac{p_0^{1/\gamma}}{\rho_0}$$

Accordingly, the initial values of all the functions can be expressed, for example, in terms of u_0 – the velocity of the gas at the axis of symmetry. In terms of u_0 we can also express the value of the derivative $dv/d\theta$ on the axis of symmetry

$$\left.\frac{dv}{d\theta}\right|_0 = -2u_0 - \frac{\rho^{\infty}(W - V_{r0}H_0)}{\rho_0 H_0}$$

By arranging the choice of the value of u_0 we can satisfy the boundary condition at the edge of the wing where $\theta = \theta_0$.

Let us consider the types of flow which arise for various values of M, α and θ_0. For the sake of simplicity we shall carry out the analysis for the case $p^{\infty} = 0$, i.e. $M = \infty$. In this case the expression to be used in the analysis for $dv/d\theta$ when $\theta = 0$ can be written in explicit form as a function of u_0, namely

$$\frac{1}{V_{\infty}}\left.\frac{dv}{d\theta}\right|_0 = \frac{1}{\cos\alpha - u^{\circ}}\left(\frac{3\gamma + 1}{2\gamma}u^{\circ 2} - 2u^{\circ}\cos\alpha + \frac{\gamma - 1}{2\gamma}\right) \tag{3.6}$$

Here $u_0 = u^{\circ}/V_{\infty}$. Let us find now the values of u° which correspond to the case $\theta_0 = \frac{1}{2}\pi$, which will be needed later.

To the relations (3.5) it is necessary in this case to add also the condition of conservation of mass at the shock, which for $\theta_0 = \frac{1}{2}\pi$ has the form

$$\rho_0 u_0 H_0 + \rho^{\infty}(W - V_{r0}H_0) = 0$$

Making use of this relation, when $M = \infty$ we find without difficulty the equation determining the values of u° for the case $\theta_0 = \frac{1}{2}\pi$

$$\frac{\gamma + 1}{2\gamma} u_{\perp}^{\circ 2} - u_{\perp}^{\circ}\cos\alpha + \frac{\gamma - 1}{2\gamma} = 0 \tag{3.7}$$

The expression so found can actually be obtained also from Formula (3.6), following from the equation of conservation of mass. For this we have to make use of the relation $dv/d\theta = -u$, valid for the translational flow with $\theta_0 = \frac{1}{2}\pi$.

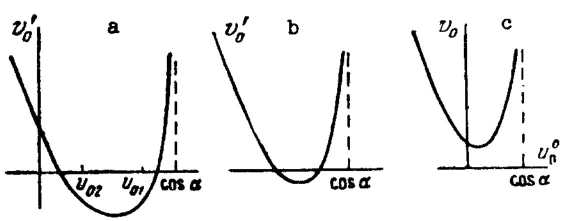

Fig. 10 a, b, c

In Fig.9 the curve 1 corresponds to the vanishing of the numerator in Expression (3.6) for $dv/d\theta$ when $\theta = 0$. On the straight line $\cos\alpha - u^{\circ} = 0$ the denominator of this expression vanishes, and curve 2 gives the value of u° when $\theta_0 = \frac{1}{2}\pi$, as a function of $\cos\alpha$, i.e. of the angle of attack.

Depending upon $\cos\alpha$ there may arise three essentially distinct cases.

a) The case $\cos\alpha > \gamma^{-1}\sqrt{\gamma^2 - 1}$. In this case when $\theta^{\circ} = \frac{1}{2}\pi$ streamline flow is possible past the wing with an attached shock. The dependence of v_0' on u_0 is depicted in Fig.10 a; the points u_{01} and u_{02}

denote the two possible values of the velocity u_0 when $\theta_0 = \frac{1}{2}\pi$. The dependence $v_\theta = -u_{01} \sin\theta$, corresponding to the larger of these two values of the velocity, is depicted by the lower curve in Fig.11 a . As u_0 increases from the value u_{01} the derivative v'_θ increases. The integral curves $v(\theta)$ are shown in Fig.11 a . This behavior is similar to that which was considered in the first example. In Fig. 12 a , b, c are shown the replacement of the behaviors of flow as θ_0 decreases from $\frac{1}{2}\pi$ to 0 . As u_0 changes from u_{01} in the direction of smaller values, the angle θ_0 at first grows from $\frac{1}{2}\pi$ to a certain limiting value, and then decreases, reaching the value $\frac{1}{2}\pi$ again when $u_0 = u_{02}$, which corresponds to the second possible behavior of flow past the wing with $\theta_0 = \frac{1}{2}\pi$ (with a stronger shock). Further decrease of u_0 leads again to a decrease in θ_0 .

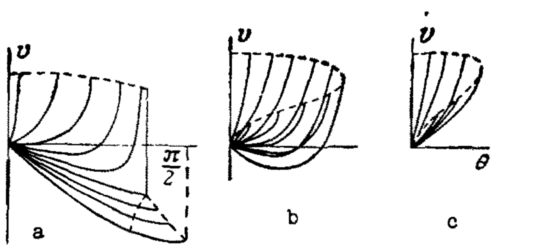

Fig. 11 a, b, c

The analysis of the flow with $u_{02} < u_0 < u_{01}$ requires the introduction of shocks inside the region of gas flow. Having regard to the limited interest of this case and the case $u_0 < u_{02}$, we shall not consider them in more detail.

b) The case $\frac{1}{2}\gamma^{-1}\sqrt{(3\gamma+1)(\gamma-1)} < \cos\alpha < \gamma^{-1}\sqrt{\gamma^2-1}$. In this case the dependence of v_0' on u_0 has qualitatively just the same form as before (Fig.10 b) but the flow past the wing with attached shock is impossible. As u_0 decreases from $V_\infty \cos\alpha$ the derivative v_0' decreases from ∞ to 0 , becomes negative, reaches a minimum, and then inctreases again. The curves $v(\theta)$ are shown in Fig.11 b. For each θ_0 we obtain two solutions and moreover there is a greatest value of θ_0 for which conical flow is still possible. The possible behaviors of flow correspond to those shown in Fig.12 b and c.

c) The case $\cos\alpha < \frac{1}{2}\gamma^{-1}\sqrt{(3\gamma+1)(\gamma-1)}$. The dependence of v_0' on u_0 is depicted in Fig.10 c, and the curves $v(\theta)$ in Fig.11 c. In this case only one behavior of flow is possible, in which all the streamlines are directed from the axis of symmetry outwards across the edge of the wing. We notice only that as $\cos\alpha$ decreases the values of u_0 become negative, i.e. the stream becomes directed towards the vertex of the wing.

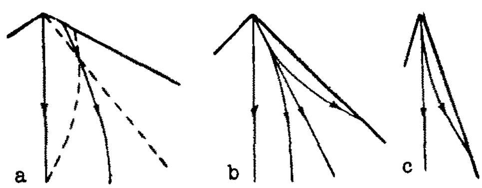

Fig. 12 a, b, c

In Fig.13, relating to the case $M = \infty$ and $\gamma = 1.4$, lines are plotted dividing the regions of different forms of symmetrical flow past a triangular wing. In the regions 1 to 4 conical behaviors of flow are possible. Moreover to each pair of values of the angle of attack α and the semiangle at the vertex of the wing θ_0 there correspond two solutions, just as for flow past a plane wing of infinite span ($\theta_0 = \frac{1}{2}\pi$) . In the regions 5 to 7 conical flow does not exist (an exception is the intersection of regions 6 and 4). In region 1 and in the parts of regions 2 and 3 to the left of the broken line the trailing edge of the wing does not have any influence upstream (in the solution with the weaker shock), and consequently solutions obtained for the infinite wing are valid also for the finite wing. In the remaining region of variation of the parameters it is necessary to consider the finite wings.

For example, at the bottom of Fig.13 we show the varaition in the pattern of flow past a wing with $\theta_0 = 10°$ with variation of the angle of attack from 0 to 180° .

In region 1 the shock is attached to the leading edges of the wing. Behind the shock there is a region of translational flow, transforming in a continuous manner (across the acoustic characteristics) into a flow, the streamlines of which asymptotically assume the direction of the center line of the wing. In the transition from region 1 to region 2 the shock becomes detached along the edge, but retains a common vertex with the wing. The velocity component normal to the edge of the wing is equal to the velocity of sound. The lines at which the velocity component vanishes and from which the stream diverges to the edges of the wing and to its center line, grow closer together as the angle of attack increases, and in the transition to region 3 (Fig.13) they merge with the center line (for wing semiangles greater than a certain $\theta_0{}^*(M)$ this merging does not occur until the conical character of the flow breaks down). In the transition to region 4 the flow of the center line of the wing changes direction: the gas begins to flow towards the vertex of the wing. For the finite wing this denotes the occurrence of a critical point on its surface. As the angle of attack increases, the conical flow becomes impossible on entry into region 5 and it is necessary to consider the finite dimensions of the wing. As the angle of attack increases further the critical point moves towards the trailing edge of the wing and leaves it. On entry into region 6 the shock becomes attached along the trailing edge, and near it there arises a region of translational flow which gradually grows and on entry region 7 it occupies the whole surface of the wing.

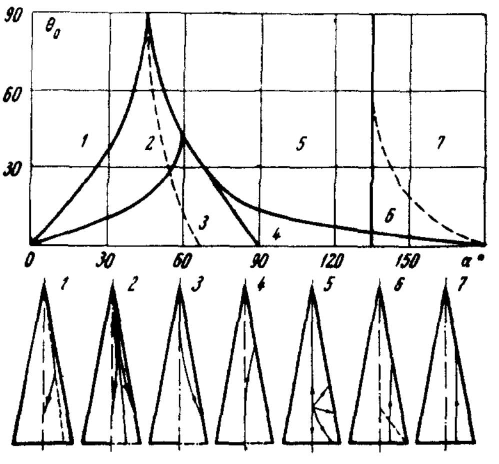

Fig. 13

With certain modifications the description of the transformations of the flow patterns applies also to wings with other vertex angles.

Let us notice a detail of fundamental interest. It was remarked above that for one and the same infinite wing, either conical flow does not exist, or else there exist two different conical flows. As is well known, in flow past a wedge ($\theta_0 = \frac{1}{2}\pi$), out of the two possible flows the one actually realized is that which corresponds to the weaker shock. The same is true also for a triangular wing with the shock attached along the edge as in region 1. But with gradual increase of the angle of attack and continuous transition from region 1 to region 4 it turns out that the stronger shock corresponds to the solution. The solution with the weaker shock now turns out to be one which is not realized. This, evidently, is the first example of two-valued steady flow with shock waves, in which we have to accord preference to the solution with the stronger shock.

In conclusion we notice that a brief derivation of the fundamental system of equations and an analysis of certain of their properties was given by the author earlier in [6 and 7].

In [8] contains examples of the calculation of flow past a triangular wing in the behavior corresponding to Fig.12 b, by using the first approximation of the method of integral relations.

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